

Hydrodynamical Description of the QCD Dirac Spectrum at Finite Chemical Potential

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(Dated: July 24, 2015)

We present a hydrodynamical description of the QCD Dirac spectrum at finite chemical potential as an incompressible droplet in the complex eigenvalue space. For a large droplet, the fluctuation spectrum around the hydrostatic solution is gapped by a longitudinal Coulomb plasmon, and exhibits a frictionless odd viscosity. The stochastic relaxation time for the restoration/breaking of chiral symmetry is set by twice the plasmon frequency. The leading droplet size correction to the relaxation time is fixed by a universal odd viscosity to density ratio $\eta_O/\rho_0 = (\beta - 2)/4$ for the three Dyson ensembles $\beta = 1, 2, 4$.

PACS numbers: 12.38Aw, 12.38Mh, 71.10Pm

1. Introduction. QCD breaks spontaneously chiral symmetry with the emergence of an octet of light mesons that permeate most of the hadronic processes at low energies [1]. Dedicated lattice simulations are now in full support of this spontaneous breaking [2]. Fundamental light quarks become constitutive and heavy producing most of the mass of the elements around us.

A remarkable feature of the spontaneous breaking of chiral symmetry is the large accumulation of the eigenvalues of the Dirac operator near zero-virtuality with the formation of a finite vacuum chiral condensate [3]. Small eigenvalue virtuality translates to large proper time, as light quarks travel very long in proper time and delocalize. The zero virtuality regime is ergodic, and its neighborhood is diffusive [4]. This behavior is analogous to disordered electrons in mesoscopic systems [5].

The ergodic regime of the QCD Dirac spectrum with its universal spectral oscillation is described by a chiral random matrix model [6]. In short, the model simplifies the Dirac spectrum to its zero-mode-zone (ZMZ). The Dirac matrix is composed of hopping between N-zero modes and N-anti-zero modes because of chirality, which are sampled from gaussian ensembles thanks to the central limit theorem. The model was initially suggested as a null dynamical limit of the instanton liquid model [7].

QCD at finite chemical potential μ is notoriously difficult to sample on a lattice due to the sign problem [8]. A number of chiral models have been proposed to describe the effects of matter in QCD with light quarks [1]. In vacuum, the chiral random matrix model simplifies the QCD

Dirac spectrum to its ZMZ. In matter, the light quark zero modes are involved. Their chiral and cross-hopping in the ZMZ is suppressed exponentially, and the corresponding Dirac matrix is banded and not random. However, large matter effects reduce the banded matrix to its diagonal, localizing the quark zero modes into molecules. In the 1-matrix model the chiral random ensemble is deformed by a constant matrix, leading to localization at large μ [9, 10]. In the 2-matrix model the deformation is still random and only generic for moderate μ with no strict banding at large μ [11, 12]. The 1-matrix approach to QCD at finite μ has been discussed by many [1, 13, 14].

In this letter we would like to combine the ergodic character of the chiral random matrix model for the low-lying modes in the ZMZ with the universal character of the hydrodynamics approach, to describe the relaxation of the QCD Dirac eigenvalues at finite μ . We will obtain the following new results: 1/ a hydrodynamical description of the Dirac eigenvalues as a droplet in the complex 2-plane; 2/ small amplitude deformations in the droplet that are gapped by the emergence of a plasmon with an odd viscosity; 3/ a non-perturbative estimate of the stochastic relaxation time for breaking/restoring chiral symmetry in matter.

2. The model. A useful model for the eigenvalues of the QCD Dirac operator at finite μ makes use of a 2-matrix model [11, 12]

$$Z_\beta[m_f] = \int \prod_{i=1}^N d^2 z_i |z_i|^\alpha \prod_{i < j}^N |z_i^2 - z_j^2|^\beta \times (z_i^2 + m_f^2)^{N_f} e^{-W(z_i)} \quad (1)$$

for quarks in the complex representation or $\beta = 2$ with

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$\alpha = \beta(\nu + 1) - 1$. ν accounts for the difference between the number of zero modes and anti-zero modes. The potential is

$$W(z) = \frac{Na\beta}{2l^2} \left(|z|^2 - \frac{\tau}{2}(z^2 + \bar{z}^2) \right) \quad (2)$$

with $l^2 \equiv 1 - \tau = 2\mu^2/(1 + \mu^2)$. For $\mu \rightarrow 0$, $\tau \approx 1$ and $l^2 \approx 2\mu^2$, so that $W(z) \approx -(N/\mu^2)(z - \bar{z})^2$, which restricts the eigenvalues to the real axis.

For the other quark representations with $\beta = 1, 4$ the joint distribution is more subtle [12]. Throughout, (1) will be assumed for $\beta = 2$, but all results extend to $\beta = 1, 2, 4$ for large N and/or the quenched approximation. All units are expressed with $a \equiv 1$ unless noted, which is related to the vacuum chiral condensate through the Banks-Casher formula [3]. Specifically $\sqrt{a} = |q^\dagger q|_0/\mathbf{n} \equiv 1$, with $\mathbf{n} = N/V_4$ the density of zero modes.

(1) can be re-written as an average of the complex fermion determinant

$$Z_\beta[m_f] = \int \prod_{i=1}^N d^2 z_i (z_i^2 + m_f^2)^{N_f} |\Psi_0[z]|^2 \quad (3)$$

using the real many-body wave-function $\Psi_0[z]$, which is the zero-mode solution to the Shrodinger equation $H_0 \Psi_0 = 0$ with the self-adjoint Hamiltonian

$$H_0 \equiv \frac{1}{2m} \sum_{i=1}^N |\partial_i + \mathbf{a}_i|^2 \quad (4)$$

Here $\partial_i \equiv \partial/\partial z_i$ and the gauge potential is $\mathbf{a}_i \equiv \partial_i S$ with $S[z] = -\ln \Psi_0[z]$. In (4) the mass parameter is $m = 1/2$.

Following [16], we observe that the Vandermonde determinant $\Delta = \prod_{i < j} |z_{ij}^2|^\beta$ induces a diverging 2-body part in H_0 . Using a similarity transformation, we can re-absorb it in $\Psi = \Psi_0/\sqrt{\Delta}$, and the new many-body Hamiltonian is

$$H = \frac{1}{\sqrt{\Delta}} H_0 \sqrt{\Delta} \quad (5)$$

We will refer to (5) as the quenched Hamiltonian. The phase quenched Hamiltonian follows a similar reasoning by rewriting (3) as

$$Z_\beta[m_f] = \int \prod_{i=1}^N d^2 z_i \left(\frac{z_i^2 + m_f^2}{\bar{z}_i^2 + m_f^2} \right)^{\frac{N_f}{2}} |\Psi_f[z]|^2 \quad (6)$$

Below we detail the hydrodynamical construction in the quenched approximation, and quote the minimal changes in the phase quenched approximation for $\beta = 2$.

3. Hydrodynamic. In the quenched approximation, we can use the collective coordinate method in [17] to

re-write (5) in terms of the density of eigenvalues as a collective variable $\rho(z) = \sum_{i=1}^N \delta^2(z - z_i)$. After some algebra and up to boundary and ultra-local terms, we obtain

$$H = \int d^2 z \rho(z) \frac{1}{2m} \left((\vec{\nabla} \pi)^2 + (\vec{\mathbf{A}})^2 \right) \equiv \int d^2 z \mathbf{h} \quad (7)$$

with the pair π, ρ canonically conjugate. We will restrict our discussion to the classical limit with the pair obeying the Poisson brackets $\{\pi(z), \rho(z')\} = \delta^2(z - z')$. Defining the even density $\rho^\chi(z) = \rho(z) + \rho(-z)$, we have

$$\vec{\mathbf{A}} = \vec{A} + \frac{1}{2} \vec{\nabla} (\beta \rho_L^\chi(z) + (\beta - 2) \ln \sqrt{\rho}) \quad (8)$$

Here ρ_L is the logarithmic transform of ρ

$$[\rho]_L \equiv \rho_L(z) = \int dz' \ln |z - z'| \rho(z') \quad (9)$$

and the vector potential ($\tau_\pm = 1 \pm \tau$)

$$\vec{A} \equiv -\frac{N\beta}{2l^2} (\tau_- x, \tau_+ y) + \frac{\alpha}{2|z|^2} (x, y) \quad (10)$$

The equation of motion for ρ yields the current conservation law and the Euler equation for \vec{v} . Defining $m\vec{v} = \vec{\nabla} \pi$, they are specifically given by

$$\begin{aligned} \partial_t \rho + \vec{\nabla} \cdot (\rho \vec{v}) &= 0 \\ \partial_t \pi + \frac{1}{2} m \vec{v}^2 + \frac{\vec{A}^2}{2m} \\ - \frac{\beta - 2}{4m\rho} \vec{\nabla} \cdot (\rho \vec{\mathbf{A}}) - \frac{\beta}{2m} \vec{\nabla} \cdot [\rho^\chi \vec{\mathbf{A}}]_L &= 0 \end{aligned} \quad (11)$$

The steady state flow from (11) corresponds to Bernoulli law with $\partial_t \pi = C$ a fixed constant. Note that all the relations hold for large but finite N , provided that the fluid density and velocity are sufficiently smooth.

4. Hydrostatic. The quenched hydrostatic solution is encoded in the condition $\mathbf{A}(z) = 0$ and $\pi = 0$. Using the formal identity $\rho_L = (2\pi/\nabla^2) \rho$, we have

$$\rho(z) = \frac{\kappa N}{\mathcal{A}} - \frac{\alpha}{2\beta} \delta^2(z) - \frac{\beta - 2}{8\pi\beta} \nabla^2 \ln \rho \quad (12)$$

where the integration constant $\kappa = 1 + \alpha/(2N\beta)$ is fixed by the density in leading order, and \mathcal{A} is the area of the eigenvalue density.

In the phase quenched approximation for $\beta = 2$, the vector potential (10) is now shifted

$$\vec{A} \rightarrow \vec{A} + \frac{N_f}{2} \vec{\nabla} \ln |z^2 + m_f^2| \quad (13)$$

with the hydrodynamical equations (11) unchanged. The corresponding phase quenched hydrostatic density (12) is modified in subleading order

$$\rho(z) \rightarrow \rho(z) - \frac{N_f}{8\pi\beta} \nabla^2 \ln|z^2 + m_f^2| \quad (14)$$

Both the quenched and phase quenched approximations describe an elliptic droplet at large N .

5. Droplet boundary. It is useful to recast the hydrostatic equations in complex form at large N

$$\begin{aligned} \bar{z} &\approx \tau z + \frac{l^2}{2N} \int_{\mathcal{A}} \frac{\rho^x(z')}{z - z'} d^2 z' \equiv S(z) \\ z &\approx \tau \bar{z} + \frac{l^2}{2N} \int_{\mathcal{A}} \frac{\rho^x(z')}{\bar{z} - \bar{z}'} d^2 z' \equiv \bar{S}(\bar{z}) \end{aligned} \quad (15)$$

$S(z)$ is a Schwartz function with $\bar{S}(S(z)) = z$ [18], i.e. $\bar{S}(z)$ is the inverse of $S(z)$. $S(z)$ is analytic outside \mathcal{A} , with $S(z) \approx \tau z + l^2/z$ asymptotically. We note the similarity with the Blue's function and its use in determining the domain boundary through conformal mapping in [13].

The support of the eigenvalues \mathcal{A} is determined by a conic curve

$$|z|^2 + C(z^2 + \bar{z}^2) + C'(z + \bar{z}) + C'' = 0 \quad (16)$$

Using the substitution $\bar{z} = S(z)$ with the asymptotic form of $S(z)$ fixes the constants in (16). The domain in $z = (x, y)$ is an ellipse $x^2/a_+^2 + y^2/a_-^2 = 1$, with the axes

$$\frac{a_{\pm}^2}{2l^2} = \frac{1 \pm \tau}{1 \mp \tau} \quad (17)$$

The ellipse remains un-split with $\mathcal{A} = \pi a_+ a_- = 2\pi l^2$. The area is preserved under τ -deformation in (2).

6. Plasmons. It is useful to analyze the small deformations in the density and velocity profile by linearizing the current conservation law in (11), i.e. $\partial_t \delta \rho + \rho_0 \nabla^2 \delta \pi = 0$, which is readily solved using

$$\delta \rho = -\rho_0 \nabla^2 \phi \quad \delta \pi = \partial_t \phi \quad (18)$$

Inserting (18) in the canonical action

$$\mathbf{S} = \int d^2 z dt (\pi \partial_t \rho - \mathbf{h}) \quad (19)$$

yields in the quadratic approximation

$$\mathbf{S} \approx \int d^2 z dt \frac{\rho_0}{2m} \left((\partial_t \vec{\nabla} \phi)^2 - W[\phi]^2 \right) \quad (20)$$

with

$$W[\phi] = \left| \vec{\nabla} \left(\frac{\beta}{2} [\delta \rho]_L^x + \frac{\beta - 2}{4} \frac{\delta \rho}{\rho_0} \right) \right|^2 \quad (21)$$

Using again the formal identity $f_L = (2\pi/\nabla^2)f$ and defining the small longitudinal field $\vec{\varphi} \equiv \vec{\nabla} \phi$, we obtain

$$\begin{aligned} \mathbf{S} \approx & N \int d^2 z dt \frac{\rho_0}{2m} \\ & \times \left((\partial_t \vec{\varphi})^2 - \left(\frac{\pi \beta \rho_0}{N} \vec{\varphi}^x + \frac{\beta - 2}{4} \nabla^2 \vec{\varphi} \right)^2 \right) \end{aligned} \quad (22)$$

after the rescaling $Nt \rightarrow t$. The small longitudinal excitations in $\vec{\varphi}$ are gapped by the plasmon frequency $\omega_p = 2\pi\beta\rho_0/N$. For an elliptic droplet of large area \mathcal{A} , (22) leads to the quadratic dispersion law

$$\omega(k) \approx \pm \left| \omega_p - \frac{\beta - 2}{4} k^2 \right| \quad (23)$$

Here $|k|$ is conjugate to $|z|$. The gapped spectrum means that the droplet is incompressible. For $\beta = 1, 2$ with quarks in the real and complex representation the branch (23) describes a plasma fluid. For $\beta = 4$ with quarks in the quaternion representation, (23) shows the start of a roton-like branch a possible indication of superfluidity. For that the higher order k^4 term is needed.

7. Odd viscosity. There is an interesting analogy between the droplet of Dirac eigenvalues at finite chemical potential, and the quantum Hall effect as a fluid of neutralized charged electrons in the plane [19, 20]. To illustrate the analogy, we first note that (10) sources the magnetic field

$$B(z) \equiv \vec{\nabla} \times \vec{A}^* = \frac{N\beta}{l^2} - \pi\alpha\delta^2(z) \quad (24)$$

with the dual notation $V_i^* = \epsilon_{ij} V_j$ subsumed. Amusingly, (4) describes a Coulomb fluid in the magnetic field (24). In large N the density of eigenvalues is uniform

$$\rho(z) \approx \frac{N}{2\pi l^2} \approx \frac{\nu B}{2\pi} \quad (25)$$

which is the density of a quantum Hall droplet with filling fraction $\nu = 1/\beta$. The plasmon frequency is the cyclotron frequency $\omega_p \equiv B/M$ with $M = N$ the analogue of the effective mass. l identifies with the magnetic length.

The k^2 -contribution in (23) is reminiscent of the odd viscosity in the fractional quantum Hall effect. To show this, let $\tilde{\pi} \equiv i\pi$ and define the collective velocity $m\tilde{v} = \vec{\nabla} \tilde{\pi} + \vec{A}^*$, then (7) is a free flow-like Hamiltonian modulo ultra-local terms,

$$H \rightarrow \int d^2z \rho(z) \frac{m}{2} \tilde{v}^+ \cdot \tilde{v} \quad (26)$$

In our case $\tilde{v}^+ \neq \tilde{v}$ but in the fractional quantum Hall effect they are equal, making \mathbf{A}^* a real gauge-field and \tilde{v} a real and gauge-invariant flow velocity for flux-riding quasi-particles [19, 20]. Assuming that $\tilde{\pi}$ and ρ are canonical and after some algebra, the Euler equation following from (26) yields the momentum conservation law

$$\partial_t(\rho m \tilde{v}_i) + \nabla_j \mathbf{T}_{ij} = 0 \quad (27)$$

with the stress tensor

$$\mathbf{T}_{ij} = m\rho \tilde{v}_i \tilde{v}_j + \frac{\beta-2}{4} \rho (\nabla_i \tilde{v}_j^* + \nabla_j^* \tilde{v}_i) \quad (28)$$

The first contribution is the classical free fluid part. The second contribution is the odd viscosity contribution following from the breaking of parity in 2-dimensions [21], with

$$\frac{\eta_O}{\rho} = \frac{\beta-2}{4} \rightarrow -\frac{1}{4}, 0, \frac{1}{2} \quad (29)$$

which is the coefficient of the k^2 term in (23). In the quantum Hall fluid, η_O originates from a mixed gauge-gravitational anomaly [22]. We note that the pair \tilde{v}, \tilde{v}^* are orthogonal. This explains that the k^2 -contribution in (23) acts as the even (shear) viscosity but without the i for dissipation. No vorticity is therefore expected.

8. Instanton and relaxation time. We identify the zero energy configuration in (7) as an instanton solution with imaginary velocity with $\pi \rightarrow i\pi$, i.e.

$$\mathbf{h} \rightarrow |\vec{\nabla}\pi|^2 - |\vec{\mathbf{A}}|^2 = 0 \quad (30)$$

that satisfies the analytically continued in time conservation law ($t \rightarrow -it_E$)

$$-\partial_{t_E} \rho + \vec{\nabla} \cdot (\rho \vec{\nabla} \pi) = 0 \quad (31)$$

Without loss of generality and for simplicity we choose $\tau = 0$ in (2) so that the hydrostatic droplet is circular. To solve (31) we set $\rho(0, z) = K/\pi \gg \rho_0$, which corresponds to all eigenvalues localized in a small disc centered around the origin to simulate a chirally restored phase at finite μ . (31) simplifies by radial symmetry

$$\begin{aligned} \partial_r \rho_L(r, t_E) &= f(r, t_E) \\ r \partial_{t_E} f + r(\beta f - \frac{N\beta r}{2l^2}) \partial_r f + f(\beta f - \frac{N\beta r}{2l^2}) &= 0 \end{aligned} \quad (32)$$

We note that similar non-linear equations emerge from the diffusion of non-hermitean matrices [23].

The solution to (32) with a free boundary or large droplet size \mathcal{A} can be obtained using the method of characteristics. Specifically

$$\begin{aligned} \frac{dt_E}{ds} &= -r \\ \frac{dr}{ds} &= -r(\beta f - \frac{N\beta r}{2l^2}) \\ \frac{df}{ds} &= f(\beta f - \frac{N\beta r}{2l^2}) \end{aligned} \quad (33)$$

with the conditions $t_E(s=0) = 0$, $r(s=0) = r_0$ and $f(s=0) = f(r_0)$. For $f(r, t_E=0) = Kr$ with $K \gg \rho_0$,

$$\begin{aligned} t_E &= -\mathbf{a}s + \frac{l^2}{N\beta} \ln \left(\frac{r_0 + \mathbf{a} - (r_0 - \mathbf{a})e^{\frac{N\beta}{l^2}\mathbf{a}s}}{2\mathbf{a}} \right) \\ r &= \mathbf{a} \frac{r_0 + \mathbf{a} + (r_0 - \mathbf{a})e^{\frac{N\beta}{l^2}\mathbf{a}s}}{r_0 + \mathbf{a} - (r_0 - \mathbf{a})e^{\frac{N\beta}{l^2}\mathbf{a}s}} \\ \mathbf{a} &= \sqrt{2Kr_0^2 l^2 / N} \\ f &= \frac{Kr_0^2}{r} \end{aligned} \quad (34)$$

We first use the three equations in (34) and solve for $r_0 = r_0(r, t_E)$. We then substitute the answer in the fourth equation in (34) to find explicitly $f(r, t_E)$ in general. Its large time asymptotic for $s \rightarrow -\infty$ is $t_E \approx -\mathbf{a}s$ and

$$f(r, t_E) \approx \frac{Nr}{2l^2} \left(\frac{1 + \sqrt{\frac{2Kl^2}{N}} - (1 - \sqrt{\frac{2Kl^2}{N}})e^{-\frac{N\beta}{l^2}t_E}}{1 + \sqrt{\frac{2Kl^2}{N}} + (1 - \sqrt{\frac{Kkl^2}{N}})e^{-\frac{N\beta}{l^2}t_E}} \right)^2 \quad (35)$$

(35) relaxes as $e^{-2\omega_p N t_E}$ to $f(r, \infty) = Nr/2l^2$ leading to the hydrostatic density $\rho_0 = N/(2\pi l^2)$. We identify $T_R \approx 1/2\omega_p$ with the relaxation time after rescaling $Nt_E \rightarrow t_E$. The scalar density relaxes by emitting two longitudinal plasmons. In physical units

$$T_R \approx \frac{1}{2\omega_p} \rightarrow \left(\frac{1 + \frac{\mathcal{A}a}{2\pi}}{2\beta} \right) \sqrt{a} \quad (36)$$

after re-instating $1 \equiv \sqrt{a} = |q^\dagger q|_0 / \mathbf{n}$, and adding the 1 to reproduce the $\mu = 0$ result in [16]. A simple extension to finite temperature amounts to a re-definition of units or $\sqrt{a} \rightarrow \sqrt{a_T} = |q^\dagger q|_T / \mathbf{n}_T$ as in [16, 24].

Finally, we note that $T_R \approx 1/2\omega_p$ translates to a diffusive time with $l^2 \equiv \mathcal{A}/2\pi \approx 2\beta T_R$. The diffusion constant is $\mathbf{D} = 2\beta$. An estimate of the finite droplet size corrections follow from (23) using the substitution $\omega_p \rightarrow \omega(k \approx 1/\sqrt{\mathcal{A}})$. The leading correction is controlled by the odd viscosity to density ratio in (29) and is small.

9. Conclusions. The hydrodynamical description captures some key aspects of the QCD Dirac eigenvalues in the diffusive regime at finite chemical potential. It supports an instanton that describes the stochastic relaxation of the Dirac eigenvalues as a fluid. The fluid is incompressible and exhibits dispersive plasmon waves that can be used to estimate the time it takes for a chirally symmetric phase to relax to a chirally broken phase in matter. The time estimate is non-perturbative and

gauge-independent.

Acknowledgements The work of YL and IZ is supported in part by the U.S. Department of Energy under Contracts No. DE-FG-88ER40388. The work of PW is supported by the DEC-2011/02/A/ST1/00119 grant and the UMO-2013/08/T/ST2/00105 ETIUDA scholarship of the (Polish) National Centre of Science.

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